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Free strings and superstrings on the $M\mathbb{C}_n$ spaces

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Abstract

We construct a free bosonic, fermionic and supersymmetric field theories on the $M\mathbb{C}_n$ spaces for critical dimensions, i.e., $n = 2^p$. This procedure allows us to exhibit the underlying geometry of these spaces. In particular, these spaces are shown to be in correspondence with the n real spheres S^{n-1}. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The complex numbers are, besides their purely theoretical aspect, one of the basic tools used to describe many of the models such as two-dimensional conformal field theories [14], string theories [5] and many other fields. It is then known that because of the no-go theorem due to Hurwitz [15,16], only four integer and composition algebras can be constructed. However and by relaxing at least one of the two conditions of validity of the Hurwitz theorem, some extensions [17–19] of complex numbers can be considered. Here we will consider the one consisting in the introduction of a fundamental unit e satisfying $e^n = -1$. The set of these numbers has been intensively studied in the last few years. The obtained algebra, called multicomplex algebra [1,2] and denoted $M\mathbb{C}_n$ is an n-dimensional $\mathbb{R}$-algebra given by

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$$M\mathbb{C}_n = \left\{ z = \sum_{i=0}^{n-1} x_i e^i, \quad x_i \in \mathbb{R} \right\};$$

more recentely, known models such as Liouville, sine-Gordon and deformed-Toda models have been studied in the context of multicomplex algebras [12,13]. In the present paper, we review [4] the case of a free bosonic field theory on the $M\mathbb{C}_n$ spaces, as well as the free fermionic and sypersymmetric cases.

The paper is organized as follows. In Section 2, we give a brief review of the main properties of multicomplex numbers. For more details, see [1,2]. We also give an equivalent version to the notion of holomorphic functions on $M\mathbb{C}_n$ in terms of truncations of the "Laplacian" operator. These truncations will be useful in the study of the fermionic model. In Section 3, we review some of the results of Ref. [4], and develop the geometric picture of the $M\mathbb{C}_n$ spaces, in particular we obtain "projections" $\pi_n : S^{n-1} \to M\mathbb{C}_n, n \geq 4$ where S^{n-1} denotes the $(n-1)$-sphere. We also obtain the length $d\tilde{s}^2$ on $M\mathbb{C}_n$ and consider the corresponding real 2-form.

In Section 4, we consider first the building of free fermionic field theory. This is obtained with the obtention of a like "Dirac operator" generalizing the known one is dimension 2. Next, we consider altogether the bosonic and fermionic models, following Ref. [20]. The obtained model is then discussed in the context of the superspace structure we developed in Ref. [7] on the $M\mathbb{C}_n$ spaces. We note here, that we restrict ourselves to the case of $n = 2^p$, these dimensions have been shown to be more interesting for many reasons [1,2,4,7].

In the end we give a summary and a commentary of the obtained results. In particular, we point out that these are the first steps to be considered.

2. General

In this section we introduce the set of multicomplex numbers and certain of their properties which are useful for the sequel.

Following Fleury et al. [1,2], consider a system of complex numbers generated by a fundamental unit e satisfying the basic relation

$$e^n = -1, \quad n \in N^*. \tag{2.1}$$

This system has been already considered by Wierstrass [3]. The obtained algebra, known as multicomplex algebra [1,2] and denoted $M\mathbb{C}_n$, is an n-dimensional $\mathbb{R}$-algebra generated by the free family $\{1, e, \ldots, e^{n-1}\}$:

$$M\mathbb{C}_n = \left\{ z = \sum_{i=0}^{n-1} x_i e^i, \quad x_i \in \mathbb{R} \right\}. \tag{2.2}$$

A faithful matrix representation of the unit e is given by the $n \times n$ diagonal matrix $(E_{ij})_{1\leq i,j \leq n}$

$$E_{ij} = k^{2i-1}\delta_{ij}, \quad i, j = 1, \ldots, n, \tag{2.3}$$

where $k = \exp(i\pi/n)$.

Given two multicomplex numbers $x = \sum_{i=0}^{n-1} x_i e^i$ and $y = \sum_{i=0}^{n-1} y_i e^i$ their product is

$$xy = \sum_{i=0}^{n-1} t_i e^i,$$

where

$$t_i = \sum_{j=0}^{n-1} x_j y_{i-j} sg(i-j), \quad i = 0, 1, \dots, n-1, \tag{2.4}$$

with $sg(i-j)$ the usual sign function equal to 1 when $i - j \geq 0$ and -1 otherwise. The algebra $M\mathbb{C}_n$ was provided by a pseudo-norm

$$\|z\|^n = \det \begin{pmatrix} x_0 & x_1 & & x_{n-1} \\ -x_{n-1} & x_0 & & \\ & \ddots & \ddots & x_1 \\ -x_1 & -x_{n-1} & & x_0 \end{pmatrix} = \det \Delta(z). \tag{2.5}$$

Notice that from general properties of circulant determinants, as $\Delta(z)$ is a nth order anti-circulant matrix, one has

$$\det \Delta(z) = \prod_{p=1}^{n} \overset{[p]}{x} \tag{2.6}$$

with the definition

$$\overset{[p]}{x} = \sum_{i=0}^{n-1} x_i k^{i(2p-1)}, \quad p = 1, 2, \dots, n. \tag{2.7}$$

One may hope then to write an expression of $\|z\|$ making the only use of $M\mathbb{C}_n$, i.e., to get a formula of $\|z\|$ using the basis vectors e^i instead of k.

This is possible if e and k have the same algebraic properties, i.e. the polynomial

$$P(x) = x^n + 1,$$

which has, besides the usual complex solutions, the multicomplex roots

$$\alpha_i = e^{2i-1}, \quad i = 1, \dots, n, \tag{2.8}$$

can be written as

$$P(x) = \prod_{i=1}^{n} (x - e^{2i-1}) = \prod_{i=1}^{n} (x - k^{2i-1}), \tag{2.9}$$

meaning hence that e and k satisfy the same relation between coefficients and roots. Actually, it has been shown [1,2] that Eq. (2.9) holds iff the dimension n of $M\mathbb{C}_n$ takes the form

$n = 2^p$. These $M\mathbb{C}_n$ spaces have then a special status, one define the pth conjugate of a multicomplex number z as follows:

$$\overset{(p)}{z} = \sum_{i=0}^{n-1} x_i e^{i(2p+1)}, \quad p = 0, 1, 2, \ldots, n-1. \tag{2.10}$$

In this case, we have then the formula

$$\|z\|^n = \prod_{p=0}^{n-1} \overset{(p)}{z}. \tag{2.11}$$

Moreover, one can verify that the conjugation, Eq. (2.10), fullfils

$$\overset{(p+n)}{z} = \overset{(p)}{z}, \qquad \overset{(p)}{z_1}\overset{(p)}{z_2} = \overset{(p)}{\overline{z_1 z_2}}, \tag{2.12}$$

we note here that in the general case, i.e., for arbitrary n, one can rewrite the expression, Eq. (2.5), as

$$\|z\|^n = \prod_{p=0}^{n-1} \sum_{i=0}^{n-1} x_i \omega^{ip} e^i, \tag{2.13}$$

where $\omega = \exp(2i\pi/n)$. But actually we have complex coefficients in the expansion of $\overset{(p)}{z}$ $(p \neq 0)$ instead of real ones in the $n = 2^p$ case (see Eq. (2.10)).

In all what follows, we restrict ourselves to the case of critical dimensions, i.e. $n = 2^p$. The notion of pth conjugate allows us to see any element z of $M\mathbb{C}_n$ as parametrized by the n multicomplex numbers $\overset{(p)}{z}$. This is equivalent to say, indeed, that Eq. (2.10) can be inverted. As an easy example, for $n = 2$, a complex number $z = x_0 + x_1 \cdot i$, parametrized by x_0 and x_1 can also be parameterized by the two conjugates $\overset{(0)}{z} = z$ and $\overset{(1)}{z} = \bar{z}$.

It is then convenient to introduce the differential operators

$$\overset{(p)}{\partial} = \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-ip} e^{-i} \frac{\partial}{\partial x_i}, \tag{2.14}$$

satisfying $\overset{(p)}{\partial}\overset{(k)}{z} = \delta^{kp}$. These operators generalize the known ones on the $M\mathbb{C}_2 = \mathbb{C}$ space, i.e. $\overset{(0)}{\partial} = \partial_z$ and $\overset{(1)}{\partial} = \partial_{\bar{z}}$. It was shown [1] that most of the theorems of complex analysis can be extended to the $M\mathbb{C}_n$, $n > 2$, spaces. A main property, which will be useful for the sequel is the notion of derivation in $M\mathbb{C}_n$.

A mapping $F : M\mathbb{C}_n \to M\mathbb{C}_n$ is completely known by its compenents on the basis e^i

$$F(z) = \sum_{i=0}^{n-1} f_i(x_0, x_1, \ldots x_{n-1}) e^i, \tag{2.15}$$

where f_i are n mapping of $\mathbb{R}^n$ to $\mathbb{R}$. It has been proved [1,2] that F is derivable at z iff the components of F satisfy

$$\partial_0 f_k(x_0, x_1, \ldots, x_{n-1}) = \mathrm{sg}(p-m)\partial_p f_m(x_0, x_1, \ldots, x_{n-1}), \quad p - m \equiv k (\mathrm{mod}\, n) \tag{2.16}$$

or equivalently iff

$$\overset{\partial}{(p)}\overset{k}{(F)} = 0, \quad \forall p \neq k \tag{2.17}$$

where $\overset{F}{(p)}$ is the kth conjugate of F which will be called a holomorphic function. A harmonic function F is defined as a holomorphic one satisfying $\Delta F = 0$, whee the "Laplacian" Δ is given by

$$\Delta = \overset{(0)}{\partial}\overset{(1)}{\partial} \cdots \overset{(n-1)}{\partial} \tag{2.18}$$

we remark that Eqs. (2.16) are in fact a generalization of the well-known Cauchy–Riemann conditions defining a holomorphic function on C. In that case a (anti-) holomorphic function $f: C \to C$ satisfies $(\overline{\partial} f = 0)\partial f = 0$ where $\overline{\partial} = \partial_{\overline{z}}$ and $\partial = \partial_z$. These conditions are particular case of Eqs. (2.17) corresponding to $n = 2$. In the case of $M\mathbb{C}_n$ spaces, $n > 2$, Eqs. (2.17) means that the holomorphy of a function F on $M\mathbb{C}_n$ is equivalent to the fact that each of its lth conjugate $\overset{(p)}{F}$ depends actually just on the pth conjugate of z, i.e. $\overset{(p)}{z}$. This feature leads us to introduce the following laplacian:

$$\Delta' = \partial_0 \partial_1 \cdots \partial_{n-1} \tag{2.19}$$

where, now, $\partial_1 = \partial / \partial \overset{(i)}{z}$ is the operator derivation in the $\overset{(i)}{z}$, $i = 0, 1, \ldots, n-1$; direction. Moreover and when constructing fermionic action on the $M\mathbb{C}_n$ space, we will see that just some truncations of operator Δ' occur, i.e., the ones given by

$$\delta_i = \partial_0 \partial_1 \cdots \check{\partial}_i \cdots \partial_{n-1}, \quad i = 0, 1, \ldots, n-1, \tag{2.20}$$

with $\check{}$ meaning deletion of the operator on the ith position. Holomorphic functions $F: M\mathbb{C}_n \to M\mathbb{C}_n$ in the ith direction satisfy then

$$\delta_i F = 0. \tag{2.21}$$

As an easy example, take the case of $n = 2$, one has then $\delta_0 = \overline{\partial}$ and $\delta_1 = \partial$, it is obvious then the holomorphic functions in the usual sense are those verifying $\delta_0 f = 0$.

In the next section, we construct a "bosonic" action on the $M\mathbb{C}_n$ spaces for $n - 2^p$. This procedure will be useful to endow the $M\mathbb{C}_n$ spaces with a consistent geometry. Moreover, the obtained lagrangian generalizes the known one on $M\mathbb{C}_2 = C$ and leads to a symmetry algebra consisting of n-commuting copies of the virasoro algebra.

3. Bosonic model

We start by giving a brief review, a two-dimensional bosonic field theory as these tools will be useful for the sequel. We note that we will restrict ourselves to the $M\mathbb{C}_n$ spaces with $n = 2^p$. As seen in Section 2, in that case each multicomplex number z can be seen as parametrized by its p-conjugates $\overset{(p)}{z}$, $p = 0, 1, \ldots, n-1$. Recall that the action describing a two-dimensional free bosonic field theory, say a bosonic string theory, is simply proportional to the area of the world sheet, Mathematically, one formula for the area of a sheet embedded in a Minkowski space of dimension D is [5]

$$S_1 \sim \int \mathrm{d}\sigma\, \mathrm{d}\tau [\dot{X}^2 X'^2 - (\dot{X}X')^2]^{1/2},$$
$$\dot{X}^\mu = \frac{\partial}{\partial\tau} X^\mu(\sigma, \tau), \quad X'^\mu = \frac{\partial}{\partial\sigma} X^\mu(\sigma, \tau), \qquad \mu = 0, 1, 2, \ldots, (D-1), \tag{3.1}$$

with $X^\mu(\sigma, \tau)$ specifying the position of a string at given values of σ, τ; the local coordinates on the world sheet. However, and due to the higher non-linearity of the action S_1, the more convenient form is, for flat Minkowski space.

$$S_2 \sim \int \mathrm{d}^2\sigma\, h^{1/2} h^{\alpha\beta} \partial_\alpha X \partial_\beta X, \quad \mathrm{d}^2\sigma = \mathrm{d}\tau\, \mathrm{d}\sigma, \tag{3.2}$$

where $h_{\alpha\beta}$ is the metric tensor of the string world sheet.

As well known [5], and in the complex notations $z = \tau + i\sigma$; $\bar{z} = \tau - i\sigma$; $\partial = \frac{1}{2}(\partial_\tau - i\partial_\sigma)$ and $\bar{\partial} = \frac{1}{2}(\partial_\tau + i\partial_\sigma)$, the two-dimensional free bosonic theory is described by the following action:

$$S_{M\mathbb{C}_2} \sim \int \mathrm{d}z\, \mathrm{d}\bar{z} g^{\alpha\beta} \partial_\alpha X \partial_\beta X, \tag{3.3}$$

with $g^{z\bar{z}} = g^{\bar{z}z} = \frac{1}{2}$ and $g^{zz} = g^{\bar{z}\bar{z}} = 0$; or equivalently

$$(g^{\alpha\beta}) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{3.4}$$

The equation of motion shows that the field $X(z, \bar{z})$ splits into two parts

$$X(z, \bar{z}) = x(z) + \bar{x}(\bar{z}) \tag{3.5}$$

such that

$$\bar{\partial} x(z) = \partial \bar{x}(\bar{z}) = 0.$$

Notice that in terms of real fields the action $S_{M\mathbb{C}_2} = S_\mathbb{C}$ is proportional to

$$S_{\mathbb{R}^2} \sim \int \mathrm{d}\sigma\, \mathrm{d}\tau [(\partial_0 X)^2 + (\partial_1 X)^2]. \tag{3.6}$$

The latter relation can be generalized to the $\mathbb{R}^n$ case, this form will be useful for building a free bosonic theory on the $M\mathbb{C}_n$, $n = 2^p$, spaces. Actually the generalization of Eq. (3.6), to an extended object, will be of the form

$$S_{\mathbb{R}^n} \sim \int \mathrm{d}\sigma^2\, \mathrm{d}\sigma^1 \cdots \mathrm{d}\sigma^{n-1} [(\partial_0 X)^2 + \cdots + (\partial_{n-1} X)^2]. \tag{3.7}$$

Notice that all previous actions given here are in fact highly connected with the geometry of the world sheet and some algebra invariance of the considered theory. These symmetries are mainly related to the principles of the covariance of the theory. So, the building of a bosonic action on the $M\mathbb{C}_n$ spaces, is, by the way, an attempt to discussion of a plausible geometry on $M\mathbb{C}_n$. This question was mentionned in [1,2].

Now, we are in a position to generalize the action Eq. (3.3) to the $M\mathbb{C}_n$ case. So, let

$$X(z) = X(\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}),$$

a field on $M\mathbb{C}_n$. The most general form of bosonic action on $M\mathbb{C}_n$ can be written as follows:

$$S^B_{M\mathbb{C}_n} \sim \int \mathrm{d}^n s\, J^{rs} \partial_r X \partial_s X, \tag{3.8}$$

where J is some tensor generalizing the one of Eq. (3.3) and $\mathrm{d}^n s$ is some integration measure on $M\mathbb{C}_n$. Consequentely, in total, we have to determine the J's components as well as the integration measure $\mathrm{d}^n s$. The admissible measure $d^n s$ we will deal with are the ones which reduce to the usual n-volume form of $\mathbb{R}^n$. This, in general, we have

$$\mathrm{d}^n s = g_{i_0 i_1 \ldots i_{n-1}}\, \mathrm{d}\overset{(i_0)}{z} \wedge \mathrm{d}\overset{(i_1)}{z} \cdots \wedge \mathrm{d}\overset{(i_{n-1})}{z}. \tag{3.9}$$

Similarly, the components J_{rs} of J will be taken so that the quantity $J^{rs}\partial_r X \partial_s X$ on $M\mathbb{C}_n$ reduces to $\sum_{i=0}^{n-1} (\partial_i X)^2$ on $\mathbb{R}^n$. It is then expected that the tensor J reflect the geometric aspect of $M\mathbb{C}_n$. In fact, these two "contraints" are some how natural if one thinks of the $M\mathbb{C}_n$ spaces as a way of complexification of $\mathbb{R}^n$. Actually, by using the faithful matrix representation of the unit element given by Eq. (2.3), one can prove the following relations [4]:

$$\mathrm{d}\overset{(i_0)}{z} \wedge \mathrm{d}\overset{(i_1)}{z} \wedge \cdots \wedge \mathrm{d}\overset{(i_{n-1})}{z} = T^{i_0 i_1 \cdots i_{n-1}}_{r_0 r_1 \cdots r_{n-1}}\, \mathrm{d}\sigma^{r_0} \wedge \mathrm{d}\sigma^{r_1} \wedge \cdots \wedge \mathrm{d}\sigma^{r_{n-1}}. \tag{3.10}$$

$$(g_{i_0 i_1 \cdots i_{n-1}})_{rk} (T^{i_0 i_1 \cdots i_{n-1}}_{r_0 r_1 \cdots r_{n-1}})_{kl} = \delta_{rl}, \quad r, 1 = 0, 1, 2, \ldots, n-1, \tag{3.11}$$

$$\sum_{r,l=0}^{n-1} J^{rl} \omega^{-ir-jl} e^{-(i+j)} = \delta_{ij}, \quad i, j = 0, 1, 2, \ldots, n-1. \tag{3.12}$$

The values $i = j = n/2$ as n is even, lead to the constraint

$$\sum_{r,p=0}^{n-1} J^{rp} (-1)^{r+p} = -1, \tag{3.13}$$

which has a particular solution given by

$$(J^{rp}) = \begin{pmatrix} 0 & & & 1/n \\ & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1/n & & & 0 \end{pmatrix}. \tag{3.14}$$

Notice then, when restricted to the case $n = 2$ in Eq. (3.13), one gets the matrix already encountered in Eq. (3.4), so for the tensor J may be related to a geometric structure on the $M\mathbb{C}_n$ spaces.

Actually, for the particular solution Eq. (3.13), the bosonic action we are looking for Eq. (3.8) reads as

$$S^B_{M\mathbb{C}_n} \sim \int \mathrm{d}\overset{(0)}{z} \wedge \mathrm{d}\overset{(1)}{z} \wedge \cdots \wedge \mathrm{d}\overset{(n-1)}{z}\ J^{rp}\partial_r X \partial_p X, \tag{3.15}$$

for which the equation of motion is

$$J^{kp}\partial_k\partial_p X = 0. \tag{3.16}$$

Again a particular solution of Eq. (3.15) is given by

$$X(\overset{(0)}{z}, \overset{(1)}{z}, \ldots, \overset{(n-1)}{z}) = \overset{(0)}{x}(\overset{(0)}{z}) + \overset{(1)}{x}(\overset{(1)}{z}) + \cdots + \overset{(n-1)}{x}(\overset{(n-1)}{z}), \tag{3.17}$$

generalizing then obviously Eq. (3.5) corresponding to the case of $n = 2$. Hence the full machinery of two-dimensional conformal field theory can be developed on $M\mathbb{C}_n$.

In the general case, Eq. (3.12) gives for distinct values of i and j the following contraints:

$$\sum_{r,p=0}^{n-1} J^{rp}\omega^{-ir-jp} = 0, \quad i, j = 0, 1, 2, \ldots, n-1, \ \ i \neq j. \tag{3.18}$$

For the values $i \neq j$ with $i \neq 0$ and $n/2$, Eq. (3.12) can be rewritten, after summation on i,

$$\sum_{r,p=0}^{n-1} J^{rp} \sum_{i=0}^{n-1} \omega^{-i(r+p)} = 0, \tag{3.19}$$

as we have [1,2], for even n, $\sum_{i=0}^{n-1} e^{2i} = 0$. The last equation leads to introduce the quantity given by

$$\delta^{(n)}r + p, 0 = \begin{cases} 1 & \text{if } r + p \equiv 0 (\text{mod } n), \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

Notice, by the way, that the usual kronecker symbol $\delta_{i,j}$ is nothing but $\delta^{(0)}_{i,j}$.

Eq. (3.19) leads then to the following expression:

$$\sum_{r,p=0}^{n-1} J^{rp}\delta^{(n)}_{r+p,0} = 0 \tag{3.21}$$

or equivalentely

$$\sum_{r=0}^{n-1} J^{rn-r} = 0, \tag{3.22}$$

with the identification $J^{0n} \equiv J^{00}$.

The values $i = j = 0$ give the constraint

$$\sum_{r,p=0}^{n-1} J^{rp} = 1, \tag{3.23}$$

we rewrite now, some of the constraints satisfied by the matrix J in a compact form. Let then $\Omega^{i,j}$, $i, j = 0, 1, 2, \ldots, n-1$, be the $n \times n$ matrices having the entries $\Omega^{i,j}_{rp}$ such that

$$\Omega^{i,j}_{rp} = \omega^{-ir-jp}, \quad i, j = 0, 1, 2, \ldots, n-1, \quad r, p = 0, 1, 2, \ldots, n-1. \tag{3.24}$$

One notes then that $\dagger\Omega^{i,j} = \Omega^{j,i}$ and that Eq. (3.18) reads as

$$\mathrm{Tr}\, J \cdot \Omega^{i,j} = 0 \quad \text{for } i \neq j. \tag{3.25}$$

The constraint equation (3.22) or equivalentely Eq. (3.19) can be written as follows:

$$\mathrm{Tr}\, J \cdot \widetilde{\Omega} = 0, \tag{3.26}$$

where $\widetilde{\Omega}$ stands for the $n \times n$ matrix

$$\widetilde{\Omega} = \frac{1}{n} \begin{pmatrix} 1 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 1 \\ & & & 1 & 0 \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & 0 & 1 & & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.27}$$

So that, finding the admissible tensor J is equivalent to solve simultaneously Eqs. (3.13), (3.23), (3.25) and (3.26). Observe that the particular solution, Eq. (3.14), satisfies to these constraints. Moreover, notice that as the matrix J has n^2 entries, the system of the constraint has an infinite set of solutions. We have $\frac{1}{2}(n^2 + n - 6)$ free parameters in the theory. Thus in the case of $n = 2$, the matrix J has a unique form, that given by Eq. (3.4).

Before going ahead, let us give some properties of the matrices $\Omega^{i,j}$ useful for the building of other solutions starting from a known one, for instance the matrix given by Eq. (3.14). One notes, however, that the matrices $\Omega^{i,j}$ as well as the matrix $\widetilde{Q}$ have many interesting algebraic properties which will be analyzed elsewhere.

It is then easy to see that these matrices satisfy the following relation:

$$\Omega^{i,j} \cdot \Omega^{p,q} = n\Omega^{p,j} \delta^{(n)}_{i+q,0}, \tag{3.28}$$

where the symbol $\delta^{(n)}_{i+q,0}$ is given by Eqs. (3.20).

Consider then the matrices $\hat{J}(i, j)$ given by

$$\hat{J}(i, j) = \dagger\Omega^{i,j} \cdot J \cdot \Omega^{i,j} \quad i, j = 0, 1, 2, \ldots, n-1, \quad i < j, \tag{3.29}$$

it is clear that these matrices are symmetric $\dagger\hat{J}(i, j) = \hat{J}(i, j)$ and moreover that we have

$$\mathrm{Tr}\, \hat{J}(i, j) \cdot \Omega^{p,q} = 0, \quad p, q = 0, 1, 2, \ldots, n-1, \quad p \neq q. \tag{3.30}$$

This is a direct consequence of Eq. (3.25) and Eq. (3.28). Thus, Eqs. (3.25) are invariant under the transformations, Eqs. (3.29). However, actually not all the matrices $\hat{J}(i, j)$ are solutions of the full problem, but rather just some of their combinations satisfy, moreover, to Eqs. (3.13), (3.23) and (3.26). The solutions related to the matrix, Eq. (3.14) reads, up to a factor of normalization as follows:

$$\widetilde{J}(i) = \sum_{j=0}^{n-1} \hat{J}(i, j). \tag{3.31}$$

In what follows, we consider the matrix J given by Eq. (3.14) and exhibit the underlying geometry induced on the $M\mathbb{C}_n$ spaces. Recall that each multicomplex number z is completely determined by its n conjugates $\overset{(0)}{z}, \overset{(1)}{z}, \dots, \overset{(n-1)}{z}$.

Consider then on $M\mathbb{C}_n$ the length element given by

$$ds^2 = J_{rk}\, \mathrm{d}\overset{(r)}{z}\, \mathrm{d}\overset{(k)}{z} *, \tag{3.32}$$

where * stands for the usual complex conjugation. As we have [1,2] $\overset{(p)*}{z} = \overset{(n-p)}{z}$, the relation, Eq. (3.32), can be expressed as follows:

$$\mathrm{d}s^2 = \frac{1}{n} \sum_{r=0}^{n-1} \mathrm{d}\overset{(r)}{z}\, \mathrm{d}\overset{(r+1)}{z}. \tag{3.33}$$

Notice that in the case of $n = 2$ we obtain the usual formula, i.e. $\mathrm{d}z\, \mathrm{d}\bar{z}$. Moreover, it has been shown [1,2] that the symmetry group that leaves invariant the pseudo-norm Eq. (2.11) can be identified with Z_n, the center of $SU(n)$. Thus one may consider a general form than that given by Eq. (3.33). Indeed, the relation, Eq. (3.33) and the obtained ones under the Z_n symmetry reproduce the set $\otimes_{i=0}^{n/2-1}\{x_i^2 + x_{i+n/2}^2\}$ i.e. a non-compact form of the skeleton Γ already obtained in Ref. [1,2]. However, consider the more general form given by

$$\mathrm{d}\widetilde{s}^2 = \frac{1}{n} \sum_{\substack{0 \le r \le n-1 \\ \sigma \in Z_n}} \mathrm{d}\overset{\sigma(r)}{z}\, \mathrm{d}\overset{\sigma(r+1)}{z}. \tag{3.34}$$

It is easy then to verify that in terms of the real coordinates, we have

$$\mathrm{d}\widetilde{s}^2 = \sum_{i=0}^{n-1} (\mathrm{d}x_i)^2. \tag{3.35}$$

For instance, one may then consider the associated real 2-form $\widetilde{\omega}$ given by

$$\widetilde{\omega} = \frac{i}{n} \sum_{\substack{0 \le r \le n-1 \\ \sigma \in Z_n}} \mathrm{d}\overset{\sigma(r)}{z}\, \mathrm{d}\overset{\sigma(r+1)}{z}, \tag{3.36}$$

which remains invariant under the following transformations:

$$\overset{(k)}{z} \to e^{i\varphi}\, \overset{(k)}{z}, \tag{3.37}$$

provided that the quantity

$$\sum_{\substack{0\le r\le n-1\\ \sigma\in Z_n}} z^{\sigma(r)} z^{\sigma(r+1)} \tag{3.38}$$

is constant, Now, this is nothing but the equation of the sphere S^{n-1} (with $n \geq 3$), up to a factor of normalization. Thus, the real 2- form $\widetilde{\omega}$, if seen as a real 2-form on S^{n-1}, is then the pullback of some form $\hat{\omega}$ on $M\mathbb{C}_n$, i.e. $\widetilde{\omega} = \pi_n^* \hat{\omega}$, where

$$\pi_n : S^{n-1} \to M\mathbb{C}_n \tag{3.39}$$

is a "natural" projection. Notice that such projections were expected as the transformations, Eqs. (3.37), are known as transformations of the S^{2m+1} sphere, $m \in \mathbb{N}^*$, in our notations $m = n/2 - 1 = 2^{p-1} - 1$.

Actually, one may require that the π_n maps satisfy some additive contraints such as, at least, continuity in order to endow the $M\mathbb{C}_n$ space with a consistent topology. Observe then the invariance of $\widetilde{\omega}$ under the transformations, Eqs. (3.37), leads to consider the study of hermitian metrics such as Eq. (3.34).

Therefore, the study of the projections π_n given by Eqs. (3.39) as well as the construction of real 2-form on the $M\mathbb{C}_n$ spaces needs more details. However, these results can be viewed as the first steps to deal with. These developments will be presented elsewhere.

Now, we turn to the discussion of a supersymmetric free field theory on the $M\mathbb{C}_n, n = 2^p$ spaces.

4. Supersymmetric model

We start first by the building of a fermionic field theory on the $M\mathbb{C}_n$ spaces. This will be useful when discussing the sypersymmetric model. Recall that the lagrangian describing a free massless fermionic field in two dimensions is given by

$$L_C \sim \overline{\psi}_1 \partial_z \psi_2 + \overline{\psi}_2 \partial_{\bar{z}} \psi_1, \tag{4.1}$$

where z and $\bar{z}$ stands for the complex coordinates and ψ_1, ψ_2 are the two components of the fermionic field $\underline{\psi}$:

$$\underline{\psi}(z, \bar{z}) = \begin{pmatrix} \psi_1(z, \bar{z}) \\ \psi_2(z, \bar{z}) \end{pmatrix}. \tag{4.2}$$

The equation of motion leads to the following conditions:

$$\begin{aligned} \partial_{\bar{z}} \overline{\psi}_2 &= \partial_z \overline{\psi}_1 = 0, \\ \partial_{\bar{z}} \overline{\psi}_2 &= \partial_z \overline{\psi}_1 = 0. \end{aligned} \tag{4.3}$$

Therefore, we have

$$\underline{\psi}(z, \bar{z}) = \begin{pmatrix} \psi_1(z) \\ \psi_2(\bar{z}) \end{pmatrix}. \tag{4.4}$$

The relation, Eq. (4.1), can be rewritten in a more compact form, which will be useful for the generalization to the case of the $M\mathbb{C}_n$ spaces, we have

$$L_C \sim (\overline{\psi}_1 \overline{\psi}_2) \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.5}$$

so that the matrix $\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}$ can be viewed as the "Dirac operator". Thus, the first step towards a fermionic action on $M\mathbb{C}_n$ is the finding of a "like" Dirac operator of Eq. (4.5). Now, and by taking into account the discussion given at the end of Section 2, it is clear that this operator is nothing but the following $n \times n$ operator matrix

$$D_n = \begin{pmatrix} 0 & 0 & & 0 & \delta_{n-1} \\ 0 & 0 & & \delta_{n-2} & 0 \\ & & \iddots & & \\ 0 & \delta_1 & 0 & & 0 \\ \delta_0 & 0 & & 0 & 0 \end{pmatrix}, \tag{4.6}$$

where the operators δ_1, $i = 0, 1, 2, \dots, n-1$ are those given by Eq. (2.20). In the case of $n = 2$, the operator D_2 coïncides with that given by Eq. (4.5).

Let then $\underline{\psi}(z) = \underline{\psi}(\overset{(0)}{z}, \overset{(1)}{z}, \dots, \overset{(n-1)}{z})$ a fermionic field on $M\mathbb{C}_n$, having n components $\overset{(0)}{\psi}, \overset{(1)}{\psi}, \dots, \overset{(n-1)}{\psi}$ where each component $\overset{(p)}{\psi}$ depends on all coordinates $\overset{(0)}{z}, \overset{(1)}{z}, \dots, \overset{(n-1)}{z}$. The most general form generalizing then Eq. (4.5) to the $M\mathbb{C}_n$ case reads as

$$L_{M\mathbb{C}_n} \sim \overline{\underline{\psi}} D_n \underline{\psi} \tag{4.7}$$

or equivalently, in terms of the components $\overset{(0)}{\psi}, \overset{(1)}{\psi}, \dots, \overset{(n-1)}{\psi}$ of $\underline{\psi}$

$$L_{M\mathbb{C}_n} \sim \sum_{k=0}^{n-1} \overline{\overset{(k)}{\psi}} \, \delta_{n-k-1} \overset{(n-k-1)}{\psi}. \tag{4.8}$$

It is easy then to show that the equation of motion have the following solutions:

$$\begin{aligned} &\overset{(k)}{\psi}(\overset{(0)}{z}, \overset{(1)}{z}, \dots, \overset{(n-1)}{z}) = \overset{(k)}{\psi}(\overset{(k)}{z}), \\ &\overline{\overset{(k)}{\psi}}(\overset{(0)}{z}, \overset{(1)}{z}, \dots, \overset{(n-1)}{z}) = \overline{\overset{(k)}{\psi}}(\overset{(n-k-1)}{z}), \quad k = 0, 1, 2, \dots, n-1, \end{aligned} \tag{4.9}$$

so that each component $\overset{(k)}{\psi}$ (resp. $\overline{\overset{(k)}{\psi}}$) has depend just in the $\overset{(k)}{z}$ (resp. $\overset{(n-k-1)}{z}$) variable, $k = 0, 1, 2, \dots, n-1$. Consequentely, we obtain then n-copies of "holomorphic" theories. The fermionic action on the $M\mathbb{C}_n$ spaces is given by

$$S^F_{M\mathbb{C}_n} \sim \int \mathrm{d}\overset{(0)}{z} \wedge \mathrm{d}\overset{(1)}{z} \wedge \dots \wedge \mathrm{d}\overset{(n-1)}{z} \; \overline{\underline{\psi}} D_n \underline{\psi}. \tag{4.10}$$

By using dimensional arguments one can show that the action Eq. (4.10) is a "conformal" invariant. As an easy example, take the case of $n = 4 = 2^2$, one can develop all the machinery of conformal fields theories, and in particular calculate all the components of the energy momentum tensor. The obtained tensor is then traceless and symmetric if the equations of motion are taken into account [6]. However, one notice that the square of the matrix operator D_n did not reproduce, up to a factor, the Laplacian Δ given by Eq. (2.18).

If we take into account this constraint, the leading operator may take the following form:

$$\widetilde{D}_n = \begin{pmatrix} 0 & 0 & & 0 & \partial_0 \\ 0 & 0 & & \partial_1 & 0 \\ & & \therefore & & \\ 0 & \delta_1 & 0 & & 0 \\ \delta_0 & 0 & & 0 & 0 \end{pmatrix}.$$

This form is not unique. The lagrangian, (4.7), reads as follows:

$$\widetilde{L}_{M\mathbb{C}_n} \sim \underline{\overline{\psi}}\widetilde{D}_n\underline{\psi} = \sum_{k=0}^{n/2-1} \overline{\overset{(k)}{\psi}}\, \delta_{n-k-1} \overset{(n-k-1)}{\psi} + \overline{\overset{(k)}{\psi}}\, \partial_k \overset{(n-k-1)}{\psi}$$

and leads then to the same solutions of the equation of motion by taking into account that the quantity $\overline{\underline{\psi}\,\underline{\psi}}$ is real. We are lead then to consider the operator matrix D_n, Eq. (4.6), then the one given by $\widetilde{D}_n$.

Now, we turn to the analysis of a supersymmetric model on $M\mathbb{C}_n$. Recall first that the $M\mathbb{C}_n$ spaces have already been endowed with a superspace structure. The obtained superspace [7], we denote here $SM\mathbb{C}_n(\hat{N})$, consists of points parametrized by the multicomplex numbers $\overset{(p)}{z}$ and the Grassmann coordinates $\overset{(p)}{\theta}_i$, $i = 1, 2, \ldots, N_p$; $p = 0, 1, 2, \ldots, n-1$; $N_p \in \mathbb{N}$. $\hat{N}$ refers then to the set $(N_0, N_1, N_2, \ldots, N_{n-1})$. We have

$$SM\mathbb{C}_n(\hat{N}) = \{(\overset{(p)}{z}, \overset{(p)}{\theta}_i), \quad p = 0, 1, 2, \ldots, n-1, \quad i = 1, 2, \ldots, N_p\}. \tag{4.11}$$

As an immediate consequence of this construction, the number of Grassmanian variables N_p depends on the direction $\overset{(p)}{z}$, in contrast with the usual two-dimensional case. In a previous reference [7], we have studied the simplest case of $\hat{N}$, i.e. that corresponding to the case of $N_p = 1$ in a fixed direction $\overset{(p)}{z}$, we have then obtained n copies of commuting usual $N = 1$ super-virasoro algebra, when p takes the values $0, 1, 2, \ldots, n-1$.

The model we present hereafter, consist first in considering all together the actions Eq. (3.8) (with J given by Eq. (3.14)) and Eq. (4.10) and second to give the superspace formulation of the obtained action. We have, Gervias and Sakita [20], the following lagrangian:

$$L_{M\mathbb{C}_n} = \mathcal{J}_{rs}\, \overset{(r)}{\partial} X \overset{(s)}{\partial} X - \sum_{k=0}^{n-1} \overline{\overset{(k)}{\psi}}\, \delta_{n-k-1} \overset{(n-k-1)}{\psi}, \tag{4.12}$$

where $\mathcal{J}_{rs} = \lambda J_{rs}$ with λ a parameter to be determined.

The lagrangian, Eq. (4.12), is then obtained after performing θ- integration, where $\theta's$ are the Grassmann variables in the theory. If we consider the case of the $N_p = 1$ in each direction, the total action leading to the lagrangian, Eq. (4.12), can be written as

$$S^T_{M\mathbb{C}_n} \sim \sum_{k=0}^{n-1} \int \mathrm{d}\overset{(0)}{z} \wedge \mathrm{d}\overset{(1)}{z} \wedge \cdots \wedge \mathrm{d}\overset{(n-1)}{z}\ \mathrm{d}\overset{(k)}{\theta} \sum_{p=0}^{n-1} \partial_{n-p-1}\phi_p^+ \overset{(p)}{D} \phi_p^-, \tag{4.13}$$

where the superfields $\phi_p^\pm$ are to be determined, and $\overset{(p)}{D}$ stands for the covariant derivations given by

$$\overset{(p)}{D} = \frac{\partial}{\partial \overset{(p)}{\theta}} + \overset{(p)}{\theta}\partial_p, \quad p = 0, 1, 2, \dots, n-1. \tag{4.14}$$

By writing then

$$\phi_p^+ = X - \overset{(p)}{\theta} Y_p, \quad \phi_p^- = X + \overset{(p)}{\theta} \overset{\overline{(n-p-1)}}{\psi}, \tag{4.15}$$

one finds after some calculations that the fields Y_p reads as follows:

$$Y_p = \delta_p \overset{(p)}{\psi}, \quad p = 0, 1, 2, \dots, n-1, \tag{4.16}$$

and moreover that the factor λ is equal to n, i.e. the tensor $\mathcal{J}_{rs}$ of Eq. (4.12) reads as follows:

$$(\mathcal{J}_{rs}) = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix}, \tag{4.17}$$

one can then following reference [20] develop all the calculations of the energy momentum tensor. It is expected that we will obtain a realization of n commuting copies of $N = 1$ super-virasoro algebra. The developments for explicit values of n, for instance the case of $n = 4$, are analyzed in [6].

In summary, we have considered in this paper a free bosonic field theory, the obtained action is a generalization of the well-known one on the field of complex number $\mathbb{C}$. This study leads to the obtention of projections $\pi_n : S^{n-1} \to M\mathbb{C}_n$, the analysis of these maps will be to benefit in the understanding of the geometry of the $M\mathbb{C}_n$ spaces, and may also help in the consideration of rich structure on the spheres S^{n-1} from the known [8–11] isomorphisms, for n even, of $M\mathbb{C}_n$ and the set $\oplus^{n/2}\mathbb{C}$. We notice here that we have not developed the solutions of Eqs. (3.11), this will be done for the case of $n = 4$ [6], the tensor $g_{i_0 i_1 \cdots i_{n-1}}$ can be seen to be related to the multilinear mapping associated to the pseudo-norm Eq. (2.5).

We have considered then the massless fermionic free field theory on $M\mathbb{C}_n$, and then constructed the sypersymmetric model. The latter corresponds to the case of $\hat{N} \equiv (0, \dots, 1, \dots, 0)$ with respect to the notations of reference [7].

Notice that the obtained results here are the first steps to the considered, as we did not discuss the space of the states of the models and mainly the understanding of the effect of the pseudo-norm Eq. (2.5) on these spaces. Another remark consists in comparing the contents of conformal field theories in D-dimensions (see [21] and references therein) for even D, and multiconformal theories developed on the $M\mathbb{C}_n$ spaces for $n = 2^p$.

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